Some remarks on black hole thermodynamics

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The purpose of this White Paper is to make some remarks on foundational questions concerning two thermodynamic "paradoxes" in black hole physics. The first "paradox" involves involves the thermal instability of two identical black holes placed at the centers of two identical blackbody cavities (see Figure 1), and the second "paradox" involves the apparent violation of the second law by a classical black hole in thermodynamic equilibrium with a heat bath at a finite temperature T, which can absorb incoming thermal radiation, but cannot emit outgoing thermal radiation.

In the first apparent thermodynamic "paradox," let us consider the following simple thought experiment: Two spherical blackbody cavities, which are facing with other in close proximity, contain at their centers two identical black holes with the same mass M, as illustrated in Figure 1. The event horizons of the two black holes have a radius R which is just slightly smaller than the radius of the spherical cavities. The two blackbody cavities are coupled to each other by means of two small, matching apertures of the same size and shape, so that they can come into thermodynamic equilibrium at the same temperature T with respect to each other.

Is the resulting thermodynamic equilibrium between the two identical black holes emitting blackbody radiation at the same Hawking temperature T, which are immersed in heat baths inside the two blackbody cavities at the same blackbody temperature T, stable or unstable?

In order to answer this question, let us begin with the Hawking temperature T of a zero-charge, zero-angular-momentum black hole, which is given by [1] [2]

$$k_B T = \frac{1}{2\pi} \frac{\hbar g}{c} \tag{1}$$

where k_B is Boltzmann's constant, $\hbar$ is the reduced Planck's constant, c is the speed of light, and
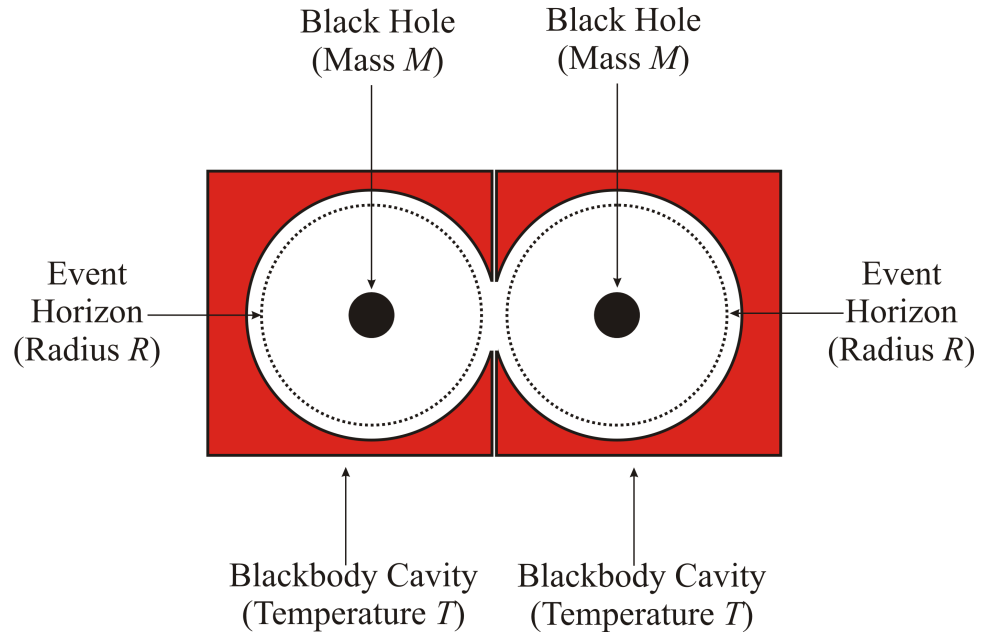
$$g = \frac{GM}{R^2} \tag{2}$$

Figure 1: Symmetrical arrangement of two blackbody cavities with two identical black holes with mass M at their centers. The two event horizons have the same radius R, which just fits inside the spherical blackbody cavities. These cavities are coupled to each other via two identical apertures, so that the two cavities come into thermodynamic equilibrium at the same temperature T. The walls of the two cavities are composed of thermally insulating material. This symmetric configuration is thermally unstable (see text).

is the "surface gravity" of the black hole, i.e., the acceleration due to gravity of a test mass located at the Schwarzschild radius R, i.e., at the event horizon of the black hole,

$$R = \frac{2GM}{c^2} \qquad (3)$$

where G is Newton's constant, and M is the mass of the black hole. Substituting (3) into (2), one finds that

$$g = \frac{c^4}{4GM} \qquad (4)$$

Substituting (4) into (1), one finds that

$$k_B T = \frac{1}{8\pi} \frac{\hbar c^3}{GM} \qquad (5)$$

This implies that

$$T \propto \frac{1}{M} \qquad (6)$$

Therefore if the mass of one of the black holes, say the one in the left blackbody cavity, were to *decrease* due to a fluctuation [3], its temperature would *increase*. However, such a temperature *increase* would result in the left blackbody cavity becoming *hotter*. This would imply an *increase* in the emission of blackbody photons from the left cavity into the right blackbody cavity. The extra photons flowing from the left cavity to the right cavity would be swallowed up by the right black hole.

Hence, by energy conservation, the *increased* rate of emission of energy by the left black hole due a fluctuation implies that its mass must *decrease*, and that the mass of the right black hole must *increase,* in a compensatory manner. But an *increase* in the mass of the right black hole would mean a *decrease* of its temperature, which would imply a further *decrease* in the mass of the left black hole, which would mean a further *increase* of its temperature, etc. This would lead to a *thermal runaway* phenomenon, i.e., an *instability* in which the system runs away from thermodynamic equilibrium. To a distant observer, the two blackbody cavities, which were originally at the same temperature, would appear to spontaneously develop a larger and larger difference in their temperatures over time.

This would seem to violate one of the common statements of the second law of thermodynamics, for example, the following textbook statement [4, p.619]:

> "The first law of thermodynamics states that energy is conserved. However, we can think of many thermodynamic processes which conserve energy but which actually never occur. For example, when a hot body and a cold body are put into contact, it simply does not happen that the hot body gets hotter and the cold body colder."

One can rephrase the last statement more precisely as follows [using the author's words]:

"It is impossible for two bodies in thermal contact with each other, which are initially in thermodynamic equilibrium at the same temperature, to spontaneously depart from this equilibrium, such that one body steadily increases in temperature and the other steadily decreases in temperature over time."

The intuition behind these putative statements of the second law is that if a movie were to be made of the behavior of the two thermally connected bodies, which could be two "black boxes" with thermometers sticking out of them in order to measure their temperatures, and with a thermally conducting strap connecting them together in order to establish thermal equilibrium, the natural "arrow of time" for the movie would be for the two black boxes to come into equilibrium at the same temperature over time. It would seem to be unnatural for the "arrow of time" to point in the opposite direction, i.e., for the two black boxes to start at the same temperature, and then spontaneously, to have one black box get progressively hotter, and the other to get progressively colder, over the course of time.

However, the above statements of the second law are false, for one can falsify them with a single counter-example, namely, two bodies undergoing a gravitational interaction with each other. Consider, for example, two identical binary stars which are initially in close proximity, so that their photospheres are "kissing" each other. Then there can arise a gravitational instability in which mass is being transferred from one star to the other. The gravitational virial theorem leads to the conclusion that the star which is losing mass will be increasing in temperature, and will become progressively brighter over time, whereas its companion, which will be gaining mass over time, will be decreasing in temperature, and will become progressively dimmer over time [5]. This counter-example demonstrates that gravitational instabilities can lead to thermal instabilities, which is consistent with the situation shown in Figure 1.

Next, let us consider the second apparent thermodynamic "paradox." "Black holes" are "black," in the sense that they are perfect absorbers of every kind of particle, including photons at all frequencies [6]. Once particles have passed through the event horizon of a black hole, they can never get out again, at a classical level of description. For in order for a particle to be able to escape from the black hole, it would need somehow to acquire an escape velocity which *effectively* (in a sense to be defined more precisely below) exceeds the speed of light at the event horizon.

Since a classical black hole is a perfect absorber, it must also be an ideal blackbody absorber that could not differ from a blackbody cavity at zero temperature. For example, a photon, once it has entered through a small aperture of an ideal, cold blackbody cavity, is irreversibly absorbed by this cavity, never again to be able to escape to infinity through this aperture (for all practical purposes). This kind of irreversible behavior of a blackbody cavity at zero temperature is no different from the irreversible behavior of a classical black hole.

However, in contrast to the case of a classical black hole, a blackbody cavity at a finite temperature must be able to *emit*, as well as to *absorb* photons [7].

Otherwise, the second law of thermodynamics would be violated. Hawking [1] suggested that, like a hot blackbody cavity, a quantum black hole would possess a finite temperature, and therefore must be able to emit particles, as well as to absorb them. But in order for a black hole to be able to emit as well as to absorb particles just like the hot blackbody cavity, there would have to be an effective modification of the no-faster-than-c particle-velocity property of the classical black hole. For only then could a particle penetrate through the event horizon with an effective escape velocity which exceeds c, and thereby be enabled to escape to infinity as a free particle.

Since such faster-than-c particle velocities at the event horizon are fundamentally impossible, classically speaking, it would be impossible for a classical black hole ever to emit any particles at all. Therefore, a classical black hole would possess an unusual kind of irreversibility, in which the black hole could capture particles, but could never release these captured particles, once they have passed through the event horizon.

However, this kind of irreversibility of a classical black hole would lead to an apparent violation of the second law of thermodynamics, since the entropy, for example, the disorder in an ambient photon "gas" contained in the thermal photons surrounding the black hole, would be swallowed up by such a classical black hole, and disappear. However, the mass of a classical black hole would not be increased by the swallowed photons, since, to an observer at infinity, these photons would have suffered an infinite gravitational redshift, and would therefore have a zero energy as they are being swallowed up. Hence there would be a decrease of the total entropy of the entire universe consisting of the unaltered mass of a classical black hole plus the altered matter and radiation outside of the black hole. In other words, the total entropy of the universe would have to decrease steadily over time as the unaltered, classical black hole steadily swallows up the thermal photons in its vicinity, but leaves no trace whatever of the disorder that was originally present in the swallowed photon "gas." This would contradict the second law of thermodynamics.

In order to "save" the second law, Bekenstein [8] had to introduce the property of the entropy of a quantum black hole, which is proportional to the area of the black hole, and to "generalize" the second law, so that the entropy of the quantum black hole must be added to the entropy of its surroundings in order to get the total entropy of the universe. In this way, the total entropy of the universe could be shown to increase, rather than to decrease, steadily over time [8][9].

In contrast to a *classical* black hole, where the event horizon has a zero width, a *quantum* black hole would possess a "fuzzy" event horizon with a nonzero width. Such quantum "fuzziness" necessarily arises from the uncertainty principle. Bekenstein conjectured that it was the Compton wavelength that was the relevant length scale for the quantum width of the event horizon. This length scale arises from a fundamental quantum uncertainty as to whether a particle of mass m (an electron, say), which has somehow been localized within a Compton wavelength of the event horizon, would be captured by the black hole, or not. When this particle is known to have been localized within the

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Compton wavelength h/mc of the horizon, its momentum must have an uncertainty of the order of $\pm mc$, i.e., it will be fundamentally uncertain whether the particle is moving towards the black hole, or moving away from the black hole, with a speed on the order of the speed of light c. Hence it will be uncertain whether a particle of mass m would be swallowed by the black hole, or would, instead, be able to escape away from the black hole as a free particle flying off to infinity with an initial speed effectively exceeding c at the event horizon [10]. Consequently, there is one bit of information when the following yes-or-no question is answered: Did this particle actually get swallowed up by the black hole, or not? This is the origin of Bekenstein's black hole entropy.

However, the localization of a particle to within a Compton wavelength of the event horizon of a black hole would also imply the possibility of pair creation and pair annihilation of particles and anti-particles near the horizon, e.g., electron-positron pair creation and pair annihilation. For if the momentum of the particle has an uncertainty of the order of $\pm mc$, the typical uncertainty in the energy of the particle will be on the order of $\pm mc^2$, which means that there would be enough energy to create pairs momentarily within the localization layer on the order of a Compton wavelength in thickness just outside the horizon, which we shall call a "Compton layer." Hence it is natural, due to these uncertainty-principle considerations, to look at pair creation and pair annihilation processes in which the net result is that, say, an electron (or a positron) could be created that could escape to infinity, and simultaneously, a positron (or an electron) could be captured by the black hole.

In fact, Hawking has pointed out one such process that could enable the penetration through the event horizon by a particle, namely, the process of quantum tunneling, which is similar to the process of the field emission of electrons from a sharp tip of a charged conductor [11]. In the case of the Hawking radiation, it is the presence of the strong tidal gravitational fields of a black hole at the event horizon, rather than of strong electric fields at the sharp tip of a charged conductor, that could tear apart virtual, fluctuating vacuum pairs, which are randomly appearing and disappearing near the horizon, into free particles that could then escape to infinity. To quote Hawking, [1, p.202]:

> "Just outside the event horizon [of a black hole] there will be virtual pairs of particles, one with negative energy and one with positive energy. The negative particle is in a region which is classically forbidden but it can *tunnel* [emphasis added] through the event horizon to the region inside the black hole where the Killing vector which represents time translations is spacelike. In this region the particle can exist as a real particle with a timelike momentum vector even though its energy relative to infinity as measured by the time translation Killing vector is negative. The other particle of the pair, having a positive energy, can escape to infinity where it constitutes a part of the thermal emission described above."

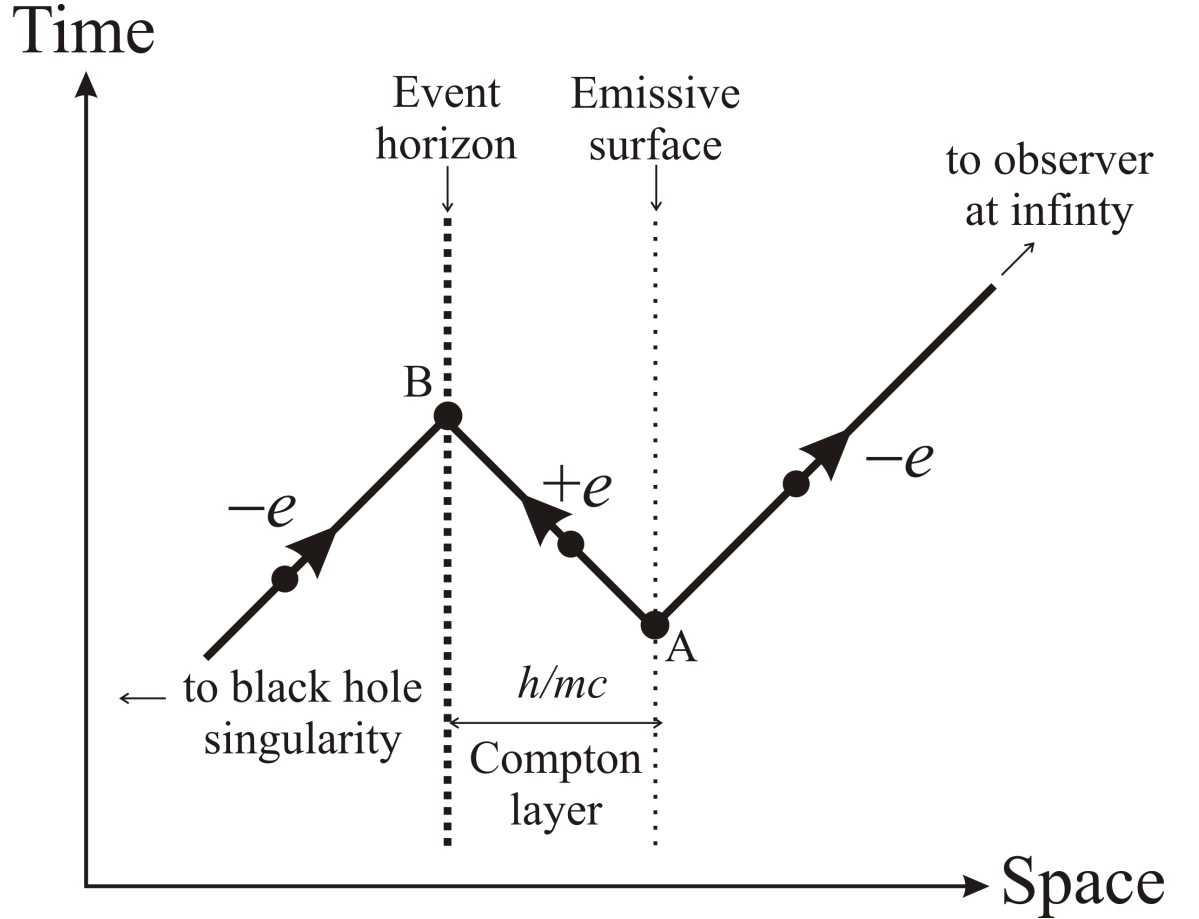The Hawking tunneling process can be understood in terms of the Feynman diagram depicted in Figure 2 [12].

Figure 2: Feynman diagram of the Hawking process for particle creation by a black hole. The heavy, zigzag solid lines represent the worldlines of electrons ($-e$) and of a positron ($+e$). The heavy, vertical dashed line on the left represents the event horizon of the blackhole, and the light, vertical dashed line on the right represents an effective emissive surface, for the case of electron emission. Points A and B represent pair creation and pair annihilation, respectively [12]. The "Compton layer" has a thickness on the order of the Compton wavelength, h/mc. The coordinate system being used here is that of a freely falling observer orbiting around the black hole just outside of its horizon.

Quantum mechanically speaking, the Hawking process could occur when a particle tunnels, apparently superluminally, through the event horizon, because its partner could then acquire, through the uncertainty principle, an *effective* escape velocity which exceeds c at the event horizon.

In particular, thermal photons could then in principle be enabled to escape to infinity from a black hole. In this way, a quantum black hole could both emit and absorb particles, just like an ideal blackbody cavity at a finite temperature, and a violation of the second law of thermodynamics could thereby be avoided.

The idea of a "superluminal tunneling" process is illustrated by the Feynman diagram in Figure 3 [12]. Feynman, in his "re-interpretation principle" [13], showed that one can re-interpret a positron going *forwards* in time as an electron going *backwards* in time. Hence in Figure 3, instead of a positron propagating forwards in time from point A to point B as in Figure 2, one can re-interpret this portion of the Feynman diagram as representing an electron propagating backwards in time from point B to point A. This can lead to an *effectively* superluminal speed for the emitted electron.

In order to see this, let us assume for the moment that the electron in Figure 3 is highly relativistic, i.e., its speed is very close to c. By inspection of Figure 3, it is clear that the electron emitted from point A would have arrived *earlier* at infinity at a detector than if it were to have been emitted from point B. Therefore if the observer at infinity had neglected the finite quantum thickness of the Compton layer in his calculation of the speed of the electron, and considered only the case of a classical black hole which could only emit electrons starting at the zero-width event horizon at point B, then it would have seemed to this observer that the emitted electron, which started from the event horizon, must have had an effective speed that exceeded c, which would be classically impossible.

But what is impossible classically may sometimes be possible quantum mechanically. Tunneling through a classically forbidden region of space is an example. Now Figure 2 is a Feynman diagram representing the Hawking tunneling process in which a positron tunnels from the *outside* to the *inside* of a black hole. But Feynman [13] showed that the two diagrams in Figures 2 and 3 are fully equivalent to each other. Hence an inverse Hawking tunneling process, in which an electron tunnels from the *inside* to the *outside* of the black hole, as represented by Figure 3, must be equivalent to the process represented by Figure 2.

The phenomenon of *superluminal* tunneling by quantum particles, including photons, which tunnel through a classically forbidden region of space with an effective group velocity which exceeds c, has been experimentally observed. The data [14] support Wigner's theory of the tunneling time [15], where it was predicted that the time it takes for a particle to traverse a tunnel barrier is independent of the thickness of the barrier, for the case of thick, opaque barriers. This counter-intuitive property of the Wigner time follows from the energy-time uncertainty principle, which, when applied to a thick, opaque tunnel barrier, implies a tunneling time that depends only on the energy of the escaping particle, but does not depend upon the distance traversed by the particle. Such a superluminal Wigner tunneling time is not forbidden by relativistic causality, because
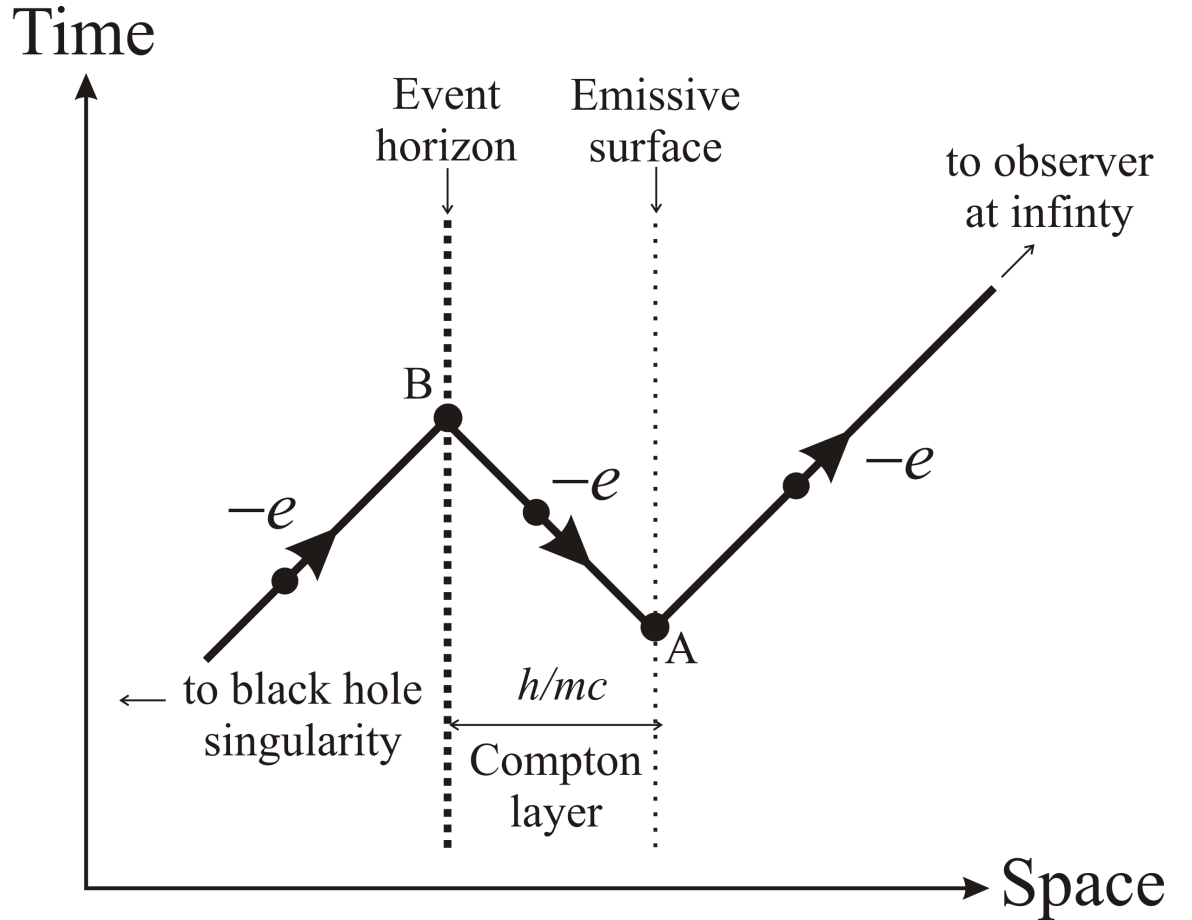
Figure 3: Equivalent Feynman diagram of the Hawking process for particle creation by a black hole. The heavy, zigzag solid line represents the worldline of a single electron $(-e)$, in which the electron is going *backwards* in time from B to A, instead of a positron $(+e)$ going *forwards* in time from A and B as in Figure 2 [12]. The resulting electron emission process is effectively superluminal (see text). It may be objected that, since the Compton layer is only h/mc thick, these superluminal effects will be tiny. However, Hawking [1] has pointed out that the Compton wavelength of a photon, which has a zero rest mass, is infinite.

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an effective group velocity for a particle can, in general, exceed c, whereas Sommerfeld's front velocity cannot [15].

Thus the process of quantum tunneling in the case of individual photons has been experimentally observed to be superluminal, in the sense that the effective group velocity of the tunneling photon exceeds c. Hence it may be possible for one member of a virtual photon pair produced in a quantum fluctuation near the event horizon of a black hole to tunnel superluminally from the outside of the event horizon to the inside, and for its partner to escape to infinity as a real photon, just like the escaping electron of Figure 2 in the Hawking tunneling process. The escaping member of the photon pair, which has been torn apart by the strong gravitational tidal forces near the event horizon of a black hole, will then appear as if it were a real blackbody photon within a Planck spectrum to a distant observer, but actually it remains in an entangled state with respect to the other member of the photon pair that was swallowed up by the black hole, just as the positron in Figure 2 that was swallowed up by the black hole will remain in an entangled state with the escaping electron.

The data [14] agreed with the prediction of the Wigner tunneling time [15]

$$\tau = \hbar \left. \frac{d\phi(E)}{dE} \right|_{E_0} = \hbar \left. \frac{d\arg(T(E))}{dE} \right|_{E_0} \tag{7}$$

where $\phi(E) = \arg(T(E))$ is the phase of the complex transfer function for the tunnel barrier, $T(E)$, and where E is the energy of the particle. The time τ is to be evaluated at the energy E_0 of the escaping particle. The physical meaning of τ is that it is the time difference between the time at which the peak of a wavepacket exits a classically forbidden region of space, and the time at which the peak of wavepacket enters this region. Wigner introduced this time, which has also been called "the group delay," in order to answer the following question: What is the delay time for the transmission of a wavepacket through a one-dimensional quantum mechanical tunnel barrier, whose transfer function $T(E)$ is known?

For thick, opaque tunneling barriers, it can be shown that this time becomes independent of the thickness of the barrier, and becomes approximately

$$\tau \simeq \hbar \frac{1}{\Delta E} \tag{8}$$

where, for the case of a rectangular tunnel barrier, ΔE is the energy difference between the height of the barrier and the energy of the incident particle. This result is consistent with the energy-time uncertainty principle, and is manifestly independent of the width of the tunnel barrier. For thick, opaque tunnel barriers, therefore, the tunneling process becomes superluminal.

There have been many theoretical "tunneling times" which have been suggested, some of which are superluminal, and others subluminal. Therefore there have been many controversies concerning which tunneling time is the "correct" one. Experiments were required to settle these controversies. In the course of performing these experiments [14], it has become clear that one must carefully

specify the operational procedure by which the "tunneling time" is actually measured.

In the case of the black hole, the operational procedure is for an observer outside of the horizon of the black hole to set up a spectrometer and detector that can detect whether a thermal photon has escaped the black hole, or not, and, if it has escaped, to measure the energy of the escaping photon. In this operational procedure, it is the superluminal Wigner tunneling time given by (7) that is the relevant tunneling time scale.

However, in the operational method that we used to measure superluminal photon tunneling times, it was crucial to be able to *compare* the two arrival times of two twin photons which were produced in a photon pair production process using parametric down conversion. The two twin photons traveled through equal distances along the two equal arms of a two-photon interferometer, i.e., the Hong-Ou-Mandel interferometer, except that one of the photons had to traverse a tunnel barrier in order to be able to reach the final beam splitter. This allowed a precise comparison of the arrival times of the two twins, one of which had tunneled through a barrier, and the other which had not. This comparison was accomplished using coincidence detection by means of two Geiger counters placed at the two output ports of the final beam splitter of the Hong-Ou-Mandel interferometer. Our results confirmed Wigner's superluminal tunneling time.

The equivalent operational method in Figure 3 would be to compare the arrival times of a photon emitted from point A with one that is emitted from point B. However, since point B lies exactly on the event horizon, it is questionable whether any emission could ever occur, since no particles, including photons, could ever escape from the horizon [16]. Hence it remains an open question whether or not the superluminal Wigner tunneling time is actually applicable to Hawking radiation from a black hole, operationally speaking.

Nevertheless, one can appeal solely to Figure 2, without any appeal to Figure 3, in order to justify the existence of Hawking radiation. Feynman's reinterpretation principle can be applied to an electron moving backwards in time, i.e, superluminally, so that the electron can be replaced with a positron moving forwards in time, i.e., without any superluminality being invoked. One could thus evade the question of whether or not the superluminality of tunneling actually occurs in black hole emission. However, this evasion would overlook the equivalence of the two Feynman diagrams shown in Figures 2 and 3.

References

[1] S.W. Hawking, "Black hole explosions?", Nature **248**, 31 (1974); "Particle creation by black holes", Comm. Math. Phys. **43** 199 (1975). A crude, order-of-magnitude estimate of the Hawking temperature can be gotten from the following argument: Imagine a small box filled with a gas of particles of mass m, which is being slowly lowered by a distant, stationary

observer, by means of a string attached to the box, towards the event horizon of a black hole. The observer stops lowering the string when the bottom of the box reaches a position just slightly on the outside of the event horizon. The box can serve as a thermometer that can measure the temperature of its surroundings. Due the gravitational attraction of the black hole, the gas inside the box, which will be in thermal equilibrium with its surroundings at a temperature T, will be denser at the bottom of the box than at its top, with a scale height H satisfying

$$k_B T \simeq mgH \qquad (9)$$

where g is the surface gravity of the black hole just outside of the event horizon. This relationship follows from the equipartition theorem. Therefore the particles of the gas will be localized to within a distance H of the bottom of the box. By the uncertainty principle, the particles near the bottom of the box will have an uncertainty in momentum on the order of

$$\Delta p \simeq \hbar/H \qquad (10)$$

Since the particles are localized near the event horizon, they will have speeds near c with an uncertainty in momentum on the order of

$$\Delta p \simeq mc \qquad (11)$$

This follows from the gravitational virial theorem. It follows that the scale height H will be on the order of

$$H \simeq \hbar/mc \qquad (12)$$

Substituting (12) into (9), one gets an approximate expression for the Hawking temperature

$$k_B T \simeq \hbar g/c \qquad (13)$$

which agrees with (1), apart from a numerical factor of $1/2\pi$. By the equivalence principle, the physics of the box suspended by the string is equivalent to that of a box undergoing a uniform acceleration a, where

$$a = g \qquad (14)$$

Hence one gets an approximate expression for the Unruh temperature

$$k_B T \simeq \hbar a/c \qquad (15)$$

which can be measured by a uniformly-accelerated box detector [2].

[2] W.G. Unruh, "Notes on black hole evaporation", Phys. Rev. D **14**, 870 (1976).

[3] Such fluctuations must necessarily arise due to the discrete nature of photons which are contained in blackbody radiation.

[4] D. Halliday and R. Resnick, *Physics* (John Wiley & Sons, New York, 1966).

[5] D. Lynden-Bell, "Negative specific heat in astronomy, physics and chemistry", arXiv:cond-mat/9812172.

[6] Another "paradox," which will not be addressed here, involves the infinite-frequency sum rule for the integral of the cross section $\sigma_{abs}(\omega)$ for the absorption of photons by a classical black hole over all frequencies ω, i.e.,

$$\int\limits_{0}^{\infty} \sigma_{abs}(\omega)d\omega = \infty \tag{16}$$

which seems to be inconsistent with the Kramers-Kronig relations and the principle of causality from which this rule is derived.

[7] This follows from Kirchhoff's law of thermal radiation.

[8] J.D. Bekenstein, "Black holes and entropy", Phys. Rev. D **7**, 2333 (1973).

[9] W.H. Zurek, "Entropy evaporated by a black hole", Phys. Rev. Lett. **49**, 1683 (1982).

[10] Note that this uncertainty-principle argument therefore also implies that a quantum black hole must be able to *emit*, as well as to *absorb*, a particle of mass m, when this particle is localized within a "fuzzy" event horizon with a quantum width of a Compton wavelength h/mc, and thereby also avoids a violation of the second law of thermodynamics.

[11] M.K. Parikh and F. Wilczek, "Hawking radiation as tunneling", Phys. Rev. Lett. **85**, 5042 (2000).

[12] It may be objected that the portion of the Feynman diagram to the left of point B in Figure 2 (and also in Figure 3) is physically meaningless, since the observer outside of the event horizon cannot see this portion of the incoming electron's trajectory, which lies in a physically inaccessible portion of spacetime. All that the outside observer can see is the sudden disappearance of the positron at point B in Figure 2 (or, by a similar argument, the sudden appearance of the electron at point B in Figure 3), and nothing more. However, the charge of the positron cannot simply disappear into nothingness at point B in Figure 2 (nor can the charge of the electron suddenly appear out of nothingness at point B in Figure 3). Otherwise, the principle of charge conservation would be violated. Thus charge conservation necessitates the annihilation of the positron's positive charge by an electron's negative charge from an unseen incoming electron which originates from beyond the event horizon in Figure 2. Moreover, the principle of momentum conservation necessitates the outgoing electron's momentum, as it emerges from point A in Figure 2, to be exactly equal to the unseen incoming electron's momentum as it approaches point B from

the left. Hence the outside observer would infer the existence of the unseen incoming portions of the Feynman diagrams which would extend beyond point B into the interior of the black hole in both Figures 2 and 3.

[13] R.P. Feynman, "The theory of positrons," Phys. Rev. **76**, 749 (1949).

[14] A.M. Steinberg, P.G. Kwiat, and R.Y. Chiao, "Measurement of the single-photon tunneling time", Phys. Rev. Lett. **71**, 708 (1993); R.Y. Chiao, P.G. Kwiat and A.M. Steinberg, "Faster than light?", Sci. Am. **269**, 52 (Aug. 1993); R.Y. Chiao and A.M. Steinberg, "Tunneling times and superluminality", in *Progress in Optics*, Vol. **37**, pp.347-406, E. Wolf (Ed.), Elsevier, Amsterdam, 1997.

[15] E.P. Wigner, "Lower limit for the energy derivative of the scattering phase shift," Phys. Rev. **98**, 145 (1955). The principle of causality used by Wigner is that the group velocity cannot exceed c, whereas the principle of causality used by Sommerfeld is that the front velocity cannot exceed c. Our experimental results [14] are consistent with Sommerfeld's principle of causality, but not with Wigner's. Nevertheless, Wigner's formula for the group delay time (7) has been verified in our experiments. It should be emphasized that Wigner's superluminal tunneling time does not violate relativistic causality because the incident wavepacket has an *analytic* waveform, for example, a Gaussian wavepacket, with a finite bandwidth (i.e., with frequencies substantially less than the height of the tunnel barrier). There exists no discontinuous "front" within a Gaussian waveform, before which the waveform is *exactly* zero. Otherwise, such a "front" would contain infinitely high frequency components that would exceed the height of the barrier, and the particle would no longer be tunneling through the barrier. The early analytic tail of the incident wavepacket, such as a Gaussian, contains all the information needed to reconstruct the entire transmitted wavepacket, including its peak, earlier in time *before* the incident peak could have arrived at a detector (for example, a Geiger counter) traveling at the speed of light. Sommerfeld first pointed out that it is the "front" velocity, and not the "group" velocity, which is forbidden by causality from exceeding c in special relativity [14].

[16] However, note that emission from point B of a photon, as well as emission from point A of a photon, is possible from a *white* hole, in contrast to the case of a *black* hole.